# Double Fell bundles over discrete double groupoids with folding

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## Abstract

In this paper we construct the notions of double Fell bundle and double C*-category for possible future use as tools to describe noncommutative spaces, in particular in finite dimensions. We identify the algebra of sections of a double Fell line bundle over a discrete double groupoid with folding with the convolution algebra of the latter. This turns out to be what one might call a double C*-algebra. We generalise the Gelfand-Naimark-Segal construction to double C*-categories and we form the dual category for a saturated double Fell bundle using the Tomita-Takesaki involution.

# 1  Introduction

Fell bundles are expected to be useful for understanding noncommutative spaces and their relationship with C*-algebras. Representations of tensor C*-categories might provide a framework for noncommutative topological quantum field theories. In a subsequent we paper we aim to test the hypothesis that a double Fell bundle tensored, or otherwise combined, with its dual be an example of a finite real spectral triple. The purpose of this paper is to work out the details of the construction of the former as well as to study Fell bundles in their own right in this category theory context.

Most of the results we present here are restricted to finite dimensions. This is partly for simplicity and partly because our motivation is to work towards constructing an example of a finite real spectral triple.

# 2  Preliminaries

## Double categories

Double categories are a sort of generalisation of 2-categories in that they are 2-dimensional categories that allow a change of boundary conditions providing a sense of 'before and after'. For this reason there are more applications to physics, where a process of change can occur.

A double category $\mathcal{DC}$ is an internal category in the category of small categories Cat[1] (Ehresman 1963). Instead of there being only arrows and objects, there are also arrows between arrows: so there is an object of objects and an object of arrows, each having arrows between them to form a category. These two types of morphism are called 1 and 2-level [15]. A double category consists of a collection of objects (or 0-cells), a collection of horizontal arrows (or horizontal 1-cells), a collection of vertical arrows (or vertical 1-cells) and a collection of 2-level arrows (or 2-cells) [4]. Objects and horizontal arrows form the horizontal category, while objects and vertical arrows form the vertical category. Each of those two categories has its identity and composition law independent of one another (that is, they satisfy the exchange or interchange law). The sources and targets of each arrow have to satisfy compatibility rules, which can be illustrated by a 'square' or 'cell':-

with the notation from [15] (For the purposes of this article with its diagrams, we swap the usual conventions corresponding to horizontal and vertical categories).

The 2-cell label is written in the centre of the square (and this may also represent the cell itself), $\alpha : M \to N$, which is not necessarily a function, for example, it might be a right module over $M$ and a left module over $N$. Cells can be composed horizontally

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[1] **Cat** is a 2-category in which, objects: small categories, morphisms: functors, 2-level morphisms: natural transformations.

$$A \xrightarrow{d(\alpha)} A'$$



Figure 1: double category cell

or vertically as decreed by the horizontal $(._h)$ and vertical $(._v)$ multiplication laws and the exchange law or interchange law: $(\alpha_1 ._v \alpha_3)._h (\alpha_2 ._v \alpha_4) = (\alpha_1 ._h \alpha_2)._v (\alpha_3 ._h \alpha_4)$ where $\alpha_1$, $\alpha_2$, $\alpha_3$, and $\alpha_4$ are cells.

Clearly, an internal category is subject to the axioms of a category; the associativity and unit laws hold strictly [1] although there exist several notions of weakening of double categories. If each category possesses all the defining properties of category '$X$', then the double category is an internal category in the category of all $X$. The compositions are defined on pairs of morphisms of the double category and one assumes that the ambient category has finite limits, so that the pullbacks exist.

A double groupoid is a small double category in which all 2-morphisms are invertible.

## Fell bundles and C*-categories

First we introduce the notion of a Fell bundle $E$ with some straightforward examples [9] and then recall the definition of a Fell bundle over a groupoid $\Gamma$. A Fell bundle over a trivial groupoid (a compact space) is the same thing as a C*-algebra bundle, and a Fell bundle over a groupoid with one object (a group) is the same thing as a $C^*$-algebraic bundle. Consider the pair groupoid $G$ on two objects. The fibre over each object is a C*-algebra: $E_{\gamma\gamma^*}$ and $E_{\gamma^*\gamma}$, and the fibre over each of the remaining two arrows $\gamma$ and $\gamma^*$ is a Banach space, $E_\gamma$ and $E_{\gamma^*}$. These two Banach spaces are modules over the said two C*-algebras. If the Fell bundle is saturated ($E_{\gamma_1}.E_{\gamma_2}$ total in $E_{\gamma_1\gamma_2}$ for all $(\gamma_1, \gamma_2) \in \Gamma^2$), they are Morita equivalence bimodules, and the linking algebra is:

$$\begin{pmatrix} E_{\gamma\gamma^*} & E_\gamma \\ E_{\gamma^*} & E_{\gamma^*\gamma} \end{pmatrix} \tag{1}$$

which is the algebra of sections of this Fell bundle.

**Definition: Fell bundle**

The definition of a Fell bundle (from [11], [9]) is recalled below:

A Banach bundle $p : E \to \Gamma$ is said to be a Fell bundle if there is a continuous multiplication $E^2 \to E$, where

$$E^2 = \{(e_1, e_2) \in E \times E \mid (p(e_1), p(e_2)) \in \Gamma^2\},$$

and an involution $e \mapsto e^*$ which satisfy the following axioms ($E_g$ is the fibre $p^{-1}(\gamma)$).

1. $p(e_1 e_2) = p(e_1)p(e_2) \ \forall \ (e_1, e_2) \in E^2$;

2. The induced map $E_{\gamma_1} \times E_{\gamma_2} \to E_{\gamma_1 \gamma_2}, \ (e_1, e_2) \mapsto e_1 e_2$ is bilinear $\forall \ (\gamma_1, \gamma_2) \in \Gamma^2$;

3. $(e_1 e_2)e_3 = e_1(e_2 e_3)$ whenever the multiplication is defined;

4. $\|e_1 e_2\| \leq \|e_1\| \|e_2\| \ \forall \ (e_1, e_2) \in E^2$;

5. $p(e^*) = p(e)^* \ \forall \ e \in E$;

6. The induced map $E_\gamma \to E_{\gamma^*}, \ e \mapsto e^*$ is conjugate linear for all $\gamma \in \Gamma$;

7. $e^{**} = e \ \forall \ e \in E$;

8. $(e_1 e_2)^* = e_2^* e_1^* \ \forall \ (e_1, e_2) \in E^2$;

9. $\|e^* e\| = \|e\|^2 \ \forall \ e \in E$;

10. $e^* e \geq 0 \ \forall \ e \in E$.

A Fell bundle whose C*-algebras are unital is a C*-category:

**Definition: C*-category**

A category with objects $A$, $B$, $C$ and homsets $(A, B)$ is a C*-category [8] if:

1. Each $(A, B)$ is a complex vector space and the composition of arrows is bilinear;

2. There is an adjoint preserving endofuctor *. (This makes $(A, A)$ into a unital *-algebra.);

3. For each $m \in (A, B)$, $m^* m$ is a positive element of the *-algebra $(A, A)$, i.e. $m^* m = y^* y$ for some $y \in (A, A)$. Furthermore $m^* m = 0$ implies $m = 0$;

4. Each $(A, B)$ is a normed space and $\|mp\| \leq \|m\| \|p\|$ for all $p \in (B, C)$;

5. Each $(A, B)$ is a Banach space;

6. For each arrow $m$, $\|m^* m\| = \|m\|^2$. (This makes $(A, A)$ into a unital C*-algebra).

# 3 Double Fell bundles

Since Fell bundles with unital C*-algebras are C*-categories, and for Fell bundles to be categories their objects must have units, we are defining what may also be called a double C*-category. We use the term double Fell bundle because one of the results of this paper is of the convolution algebra of a discrete double groupoid with folding, and it is Fell bundles rather than C*-categories that are commonly associated to groupoids in this way. There is also a motivation from gauge theory to consider a categorified Fell bundle as an analogy of a categorified fibre bundle. We coin the symbol dFB to mean something more specific:- a double Fell bundle over a double groupoid with folding, but we construct the convolution algebra explicitly only in the discrete case.

For references on double groupoids with folding structure (or connection) we refer to [10], [4], [5].

We will continually make reference to the first example constructed in 3.2 for clarification purposes. This is a double Fell line bundle over a principal, discrete double groupoid with folding consisting of just one 'square'. This will be our prototype example and we will refer to it throughout this text as 'example 1'.

## 3.1 Definition: Double Fell bundle over discrete double groupoid with folding structure

We begin by making a direct generalisation of Fell bundle over groupoid to Fell bundle over double groupoid.

1. **A Fell bundle over a double groupoid**

   Let $G$ be a double groupoid. We begin by replacing $\Gamma$ in the definition of Fell bundle $E$ over groupoid $\Gamma$ with $G$. That is to say, let $p : E \to G$ be a Banach bundle; set

   $$E^2 = \{(e_1, e_2) \ \in \ E \times E \mid (p(e_1), p(e_2)) \ \in \ G^2\},$$

   a continuous multiplication and an involution $e \mapsto e^*$ and apply the Fell bundle axioms from 1 through to 10.

   For this to make sense, the algebra of sections of a Fell *line* bundle over a double groupoid must correspond to the convolution algebra of the latter. We were able only to work out the details for this for discrete double groupoids with folding structure. Besides, if we consider the construction of example 1 to be general, in the sense that should be a method by which any double Fell bundle in the discrete case may be constructed, then they are always over double groupoids with folding because the multiplication in the definition of

Fell bundle implies that this be the case. We make this premise explicit further below on convolution algebras. Let us be clear that it may be possible to define a double Fell bundle over a double groupoid without folding structure but in this paper we define the symbol dFB to stand for a double Fell bundle over a double groupoid with folding.

## 2. Categorifying

With morphisms as the fibres (in particular the objects are the C*-algebras - which must be unital - over the groupoid units and the morphisms are the Banach spaces over the groupoid arrows) a Fell bundle $E$ is a small category with composition being tensor product. The unit law and associativity law hold only up to isomorphism, hence a bicategory. In the case of a saturated Fell bundle, each $E_g$ is an $E_{gg^*}$-$E_{g^*g}$ Morita equivalence bimodule and the Fell bundle is a small subcategory of the category of C*-algebras (objects) and imprimitivity bimodules (morphisms).

A double Fell bundle dFB over a double groupoid (with folding) is a small category internal in Bicat. As such it is weak in both directions. That is to say, in both horizontal and vertical directions the composition and unit laws hold only up to isomorphism:-

For any objects $A$, $B$ with $M \in (A, B)$ and for any 2-morphism $M, O, S$,

$$id_A \otimes M \cong M, \quad M \otimes id_B \cong M, \quad (M \otimes O) \otimes S \cong M \otimes (O \otimes S)$$

in either direction.

As it has been explained by Leinster, who writes about bicategories in [12], any weak 2-category is equivalent to a strict 2-category and so all useful calculations are expected to be equivalently carried out in a strict category counterpart. We will not attempt to develop a coherence theorem for dFB[2] but proceed as follows. For practical purposes we do not expect it to matter that dFB is doubly weak because its 'double *-algebra' (defined below) should satisfy its own exchange law nonetheless, and this we demonstrate for the discrete case. Moreover, we can replace the Banach spaces $E_g$ with their constituent vectors and we get a Banach *-category. The category axioms hold strictly, that is to say, the associativity and unit laws are equations rather than just isomorphisms. Specifically, if we let each vector in $E_g$ become a morphism in its own right then recalling the associative multiplication rule defined within Fell bundle, we get a strict category. This replacement could be formalised by defining a weak equivalence double *-functor but it would have more semantical than mathematical content.

A dFB consists of the following data:

- categories $\mathrm{Hor}_0$ and $\mathrm{Ver}_0$, which are Fell bundles as categories;

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[2]double categories that consist of a bicategory and a 2-category (weak in one direction) have been studied in [10] and [14]

- 2-morphisms $\alpha$:- These have already been defined implicitly. They are the Banach spaces over the 2-arrows in $G$. Collecting these together with $\mathrm{Hor}_0$ and $\mathrm{Ver}_0$ we have the bicategories Hor and Ver respectively.

The diagram of a 'square' or 'cell' of a dFB states the notation we will use for objects, morphisms and 2-morphisms. Although strictly speaking the term '2-morphism' refers to 0-cells and 1-cells as well as 2-cells, we will refer to them separately as objects, morphisms and 2-morphisms for clarification.

$$
\begin{array}{ccc}
A & \xrightarrow{d(\alpha)} & A' \\
\Big\downarrow{\scriptstyle M} & \alpha & \Big\downarrow{\scriptstyle N} \\
B & \xrightarrow[r(\alpha)]{} & B'
\end{array}
$$

Figure 2: double category Fell bundle cell

They are not drawn in, but there are of course for every fibre over a groupoid arrow $g$, also a fibre $E_{g^*}$ over $g^*$. Moreover, the symbol $\alpha$ actually represents four 2-morphisms, $\alpha$ from $M$ to $N$, which is also from $d$ to $r$, $\alpha^*$ from $N^*$ to $M*$ (and from $r^*$ to $d^*$), $\alpha'$ from $r$ to $d$, and $N$ to $M$, and finally $\alpha'*$ from $d^*$ to $r^*$ and $M^*$ to $N^*$. If $E$ is saturated, then $E_{g^*} \cong E_{g^*}$.

**Definition: A double C\*-category** consists of C\*-categories $\mathrm{Ver}_0$ and $\mathrm{Hor}_0$. The 2-morphisms are the completed set of linear maps from the Banach space $(A, B)$ or $M$ to the Banach space $(A', B')$ or $N$ and so on and satisfy $\| \alpha_1 \alpha_2 \| \leq \| \alpha_1 \| \| \alpha_2 \|$. Compositions are associative and if a 2-morphism is composed with the identity in the range or domain of its range or domain morphism there is no change. For example if $\alpha_a$ is a 2-morphism from $M$ to $N$, $\alpha_a id_{B'} = \alpha_a$ and $id_A \alpha_a = \alpha_a$.

3. **Double functors**

A dFB is a small category internal in Bicat while $G$ is a small category (where all 2-morphisms are isomorpisms) internal in 2-cat. As shown below, if $E$ is saturated, $p^{-1}$ is a weak double \*-functor between them. The union over all elements $g \in G$ of $p^{-1}(g)$ is not distinguishable from the dFB itself, so a saturated dFB is an internal category of functors and natural transformations. Moreover, considering the strict version of a general dFB where morphisms are $e \in E$ and multiplication is bilinear $(e_1, e_2) \in E^2$ then $p$ is a strict double \*-functor.

(a) **Definition: double functor** [13]

An internal functor or double functor between two internal categories $C$ and $D$ in the same ambient category is defined to be a pair of maps $f_0$ and $f_1$ such that the following diagrams commute:-

$$\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\ f_1 \times f_1\ } & D_1 \times_{D_0} D_1 \\
\Big\downarrow{\scriptstyle \gamma_C} & & \Big\downarrow{\scriptstyle \gamma_D} \\
C_1 & \xrightarrow{\ f_1\ } & D_1
\end{array}
\qquad
\begin{array}{ccccc}
C_1 \underset{r}{\overset{d}{\rightrightarrows}} C_0 & \xrightarrow{\ i\ } & C_1 \\
\Big\downarrow{\scriptstyle f_1} \quad \Big\downarrow{\scriptstyle f_0} & & \Big\downarrow{\scriptstyle f_1} \\
D_1 \underset{r}{\overset{d}{\rightrightarrows}} D_0 & \xrightarrow{\ i\ } & D_1
\end{array}$$

Figure 3: double functor

These diagrams state that a double functor preserves all pullbacks, identities and domain and range maps.

A weak double functor is defined by replacing the equalities from the commuting diagrams with isomorphisms. The ambient category is then Bicat.

(b) **Folding**

We define a double functor $hol_E : \mathrm{Hor}_0 \to \mathrm{Ver}$ called folding (as in [10]) such that horizontal morphisms be composable with vertical morphisms. At least in the case of example 1 3.2, this map comes with the Fell bundle by definition so we are really only identifying what already exists. Recalling that $E^2 = \{(e_1, e_2) \in E \times E \mid (p(e_1), p(e_2)) \in G^2\}$ where $G^2$ includes the extra compositions allowed due to folding. In the case of example 1, all the compositions can be illustrated by multiplying two sections (11) together.

The usual convention is to define folding to be a map from $\mathrm{Ver}_0$ to $\mathrm{Hor}$, thus it is a horizontalisation of the vertical morphisms. We use the opposite convention for the sake of clarity in the description of example 1. Note that additional compositions are included that do not occur in a double groupoid without folding. Later we further demonstrate these additional compositions in the construction of the convolution algebra. We simultaneously impose a corresponding compatible folding map on the double groupoid such that $p$ is a double *-functor as verified below. This means for example that if $N$ is composable with $d$, then the groupoid arrows $p(N)$ and $p(d)$ must be composable.

(c) **Double functor between $E$ and $G$**

With $f = p^{-1}$ (since $f_1$ determines $f_0$ we drop the subscripts) and $E$ a saturated dFB we have the following isomorphisms:

$$p^{-1}(g_1)p^{-1}(g_2) \cong p^{-1}(g_1 g_2) \tag{2}$$
$$p^{-1}(id_{gg^*}) = id_{p^{-1}(gg^*)} \tag{3}$$
$$p^{-1}(id_{g^*g}) = id_{p(g^*g)} \tag{4}$$
$$p^{-1}(d(g)) = d(p^{-1}(g)) \tag{5}$$
$$p^{-1}(r(g)) = r(p^{-1}(g)) \tag{6}$$

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for all $g, g_1, g_2 \in G$.

We add the following condition:-

$p^{-1}(g^*) = E_{g^*} \cong (E_g)^*$

to make $p^{-1}$ into an adjoint preserving functor.

Each of these is evidently satisfied by any saturated Fell bundle and therefore by any saturated dFB. (In general only $p^{-1}(g_1)p^{-1}(g_2) \in p^{-1}(g_1 g_2)$.)

Secondly, $p : E \to G$ is a strict double *-functor from strict dFB $E$ to $G$:

$$p(e_1 e_2) = p(e_1)p(e_2) \ \forall \ (e_1, e_2) \ \in \ E^2 \tag{7}$$

$$p(id_A) = id_{p(A)} \ \forall \ objects \ A \tag{8}$$

$$p(d(e)) = d(p(e)) \ \text{and} \ p(r(e)) = r(p(e)) \ \forall \ e \ \in E \tag{9}$$

$$p(e^*) = p(e)^* \ \forall \ e \ \in \ E \tag{10}$$

Where $d(e)$ is the C*-algebra that $E_g \ (\ni e)$ is a left module over, and $r(e)$ is the C*-algebra that $E_{g^*}$ is a right module over. If $E$ is saturated then $d(e)$ and $r(e)$ are the C*-algebras that $E_g$ is an imprimitivity bimodule over.

All of these are clearly satisfied by any Fell bundle and any dFB by definition.

We conclude that $p^{-1}$ is a weak double *-functor when $E$ is saturated and $p$ is a strict double *-functor for any dFB, $E$.

Since a folding is equivalent to a connection [5] we may also refer to $G$ as a double groupoid with connection. This may be a fortunate term as the multiplication $e_1 e_2$ as in Fell bundle definition provides a rule for multiplying a vector in one fibre by a vector in an adjacent fibre (wherever the double groupoid with connection pullbacks are preserved). The folding formalises some of those compositions allowed by this Fell bundle definition but we identify it as they do not take place in a general double groupoid.

4. **Convolution algebra**

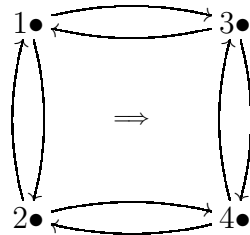Consider a 'square' of a principal, discrete double groupoid with folding, $G$:-



Figure 4: double groupoid cell

its convolution algebra is the algebra of sections of a line bundle $E$ over it. The corresponding dFB square is:-
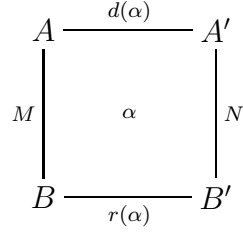
9

Figure 5: double category Fell bundle cell

where in this case each 2-cell, 1-cell and 0-cell is a copy of the complex line. Again there are also the hermitian conjugate 2-morphisms. We denote these as $M^*$, $N^*$, $d^*$, $r^*$, $\alpha^*$, $\alpha'^*$ despite the fact that the isomorphism rather than the equality holds: $E_{g^*} \cong E_g^*$ because we will be doing all calculations in terms of the algebra of sections. Using lower case letters to denote elements for example $m \in M$, and also with $\alpha_i$ to denote elements of the 2-morphisms, we write an element of the linking algebra (whose construction we explain in more detail in 3.2):-

$$
\begin{pmatrix}
a & m^* & d_1^* & \alpha_1^* \\
m & b & \alpha_1'^* & r_1^* \\
d_1 & \alpha_1' & a' & n^* \\
\alpha_1 & r_1 & n & b'
\end{pmatrix}
\tag{11}
$$

Note that the basis is given by the set of objects as in the convolution algebra for an ordinary discrete groupoid. Thus we have given the algebra of sections of a line bundle $E$ and have demonstrated the compositions that take place in $G$. For the full convolution algebra of a discrete double groupoid we need to compose adjacent squares both horizontally and vertically and for this we require a notion of double *-algebra as follows.

**Definition: A double \*-algebra** or involutive double algebra $A$ is a graded algebra with two multiplications $_h$ and $_v$ and two involutions * and †. For practical purposes let us add that it be unital and associative. [3]

(a) $A = \cup A^{\alpha_i}$

where the index set is a label on elements of some mathematical structure that will decree which compositions are allowed and which are not. For example, the squares of a double groupoid, in which case one can multiply $\alpha_i$ and $\alpha_j$ if and only if they share a common edge, be that a horizontal edge ($_v$) or a vertical edge ($_h$);

(b) $A^{\alpha_i} \cap A^{\alpha_j} = \{0\}$ for all $\alpha_i$ and $\alpha_j$ that do not compose (which in the case of the double groupoid example, this means they do not share a common edge);

(c) $a_{\alpha_i} a_{\alpha_j} \in A^{\alpha_i \alpha_j}$ for all adjacent $\alpha_i$ and $\alpha_j$;

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[3] this is based on the definition of involutive 2-algebras by Resende (for which there is currently no reference).

(d) $a_\alpha^* \in A^{\alpha*}$ for all $a_\alpha \in A^\alpha$;

(e) $A^{\alpha=0} = 0$;

(f) $(a^\dagger)^* = (a^*)^\dagger$;

(g) $a_{\alpha_i}\ _h a_{\alpha_j} = a_{\alpha_i}\ _v a_{\alpha_j}$ for all $\alpha_i, \alpha_j \in A$;

(h) $(a_{\alpha_1}\ _h a_{\alpha_2})_v(a_{\alpha_3}\ _h a_{\alpha_4}) = (a_{\alpha_1 v}a_{\alpha_3})_h(a_{\alpha_2}\ _v a_{\alpha_4})$;

(i) If the additional condition that $\| a_\alpha^* a_\alpha \| = \| a_\alpha \|^2$ is true for all $a_\alpha \in A^\alpha$ then $A$ is called a double C*-algebra.

**Example**

Let the index set $\alpha_i$ label the squares of a discrete double groupoid with folding, $G$. Then its convolution algebra $A$ is a double C*-algebra as defined above. Each $A^{\alpha_i}$ is the convolution algebra as described above for that square. Two squares can compose if and only if they share a common edge. We refer to the exchange law below because it demonstrates how the two multiplications are carried out. Both involutions are the same, that is, hermitian conjugate and so condition (f) is satisfied immediately. In this context of finite dimensional algebras, it is clear that $A$ satisfies (i) and therefore $A$ is a double C*-algebra.

The interchange law is demonstrated below diagrammatically and with matrices:

Combining two squares of dFB horizontally:



$$a_{\alpha_1} a_{\alpha_2} = \begin{pmatrix} a & m^* & d_2^* d_1^* & \alpha_2^* \alpha_1^* \\ m & b & \alpha_2'^* \alpha_1'^* & r_1^* \\ d_1 d_2 & \alpha_1' \alpha_2' & a'' & o^* \\ \alpha_1 \alpha_2 & r_1 r_2 & o & b'' \end{pmatrix} \in A^{\alpha_1 \alpha_2} \tag{12}$$

$$a_{\alpha_3}a_{\alpha_4} = \begin{pmatrix} b & p^* & d_4^*d_3^* & \alpha_4^*\alpha_3^* \\ p & c & \alpha_4'^*\alpha_3'^* & r_1^* \\ d_3d_4 & \alpha_3'\alpha_4' & b'' & r^* \\ \alpha_3\alpha_4 & r_3r_4 & r & c'' \end{pmatrix} \in A^{\alpha_3\alpha_4} \tag{13}$$

Combining the above two horizontally composed squares vertically:-



$$a_{\alpha_1}a_{\alpha_2}a_{\alpha_3}a_{\alpha_4} = \begin{pmatrix} a & p^*m^* & d_2^*d_1^* & \alpha_4^*\alpha_3^*\alpha_2^*\alpha_1^* \\ mp & c & \alpha_4'^*\alpha_3'^*\alpha_2'^*\alpha_1'^* & r_4^*r_3^* \\ d_1d_2 & \alpha_1'\alpha_2'\alpha_3'\alpha_4' & a'' & r^*o^* \\ \alpha_1\alpha_2\alpha_3\alpha_4 & r_3r_4 & or & c'' \end{pmatrix} \tag{14}$$

Now we combine vertically and then horizontally:



$$a_{\alpha_1}a_{\alpha_3} = \begin{pmatrix} a & p^*m^* & d_1^* & \alpha_3^*\alpha_1^* \\ mp & c & \alpha_3'^*\alpha_1'^* & r_3^* \\ d_1 & \alpha_1'\alpha_3' & a' & q^*n^* \\ \alpha_1\alpha_3 & r_3 & nq & c' \end{pmatrix} \in A^{\alpha_1\alpha_3} \tag{15}$$

$$
\begin{array}{ccc}
A' \xrightarrow{\;d(\alpha_2)\;} A'' & & \\
\end{array}
$$

First diagram (left):

$$
\begin{array}{ccc}
A' & \xrightarrow{d(\alpha_2)} & A'' \\
\downarrow{\scriptstyle N} \quad {\scriptstyle \alpha_2} & & \downarrow{\scriptstyle O} \\
B' & \xrightarrow[d(\alpha_4)]{r(\alpha_2)=} & B'' \\
\downarrow{\scriptstyle Q} \quad {\scriptstyle \alpha_4} & & \downarrow{\scriptstyle R} \\
C' & \xrightarrow{r(\alpha_4)} & C''
\end{array}
\qquad = \qquad
\begin{array}{ccc}
A' & \xrightarrow{d(\alpha_2)} & A'' \\
\downarrow{\scriptstyle N\otimes Q} \quad {\scriptstyle \alpha_2\,\otimes\,\alpha_4} & & \downarrow{\scriptstyle O\otimes R} \\
C' & \xrightarrow{r(\alpha_4)} & C''
\end{array}
$$

$$
a_{\alpha_2}a_{\alpha_4} =
\begin{pmatrix}
a' & q^*n^* & d_2^* & \alpha_4^*\alpha_2^* \\
nq & c' & \alpha_4'^*\alpha_2'^* & r_4^* \\
d_2 & \alpha_2'\alpha_4' & a'' & r^*o^* \\
\alpha_2\alpha_4 & r_4 & or & c''
\end{pmatrix}
\in A^{\alpha_2\alpha_4}
\qquad (16)
$$

$$
\begin{array}{ccccc}
A & \xrightarrow{d(\alpha_1)} & A' & \xrightarrow{d(\alpha_2)} & A'' \\
\downarrow{\scriptstyle M\otimes P} \;{\scriptstyle \alpha_1\otimes\alpha_3} & {\scriptstyle N\otimes} & \downarrow{\scriptstyle Q}\;{\scriptstyle \alpha_2\otimes\alpha_4} & & \downarrow{\scriptstyle O\otimes R} \\
C & \xrightarrow{r(\alpha_3)} & C' & \xrightarrow{r(\alpha_4)} & C''
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{d(\alpha_2)\,\circ\,d(\alpha_1)} & A'' \\
\downarrow{\scriptstyle M\otimes P}\;{\scriptstyle (\alpha_2\otimes\alpha_4)\circ(\alpha_1\otimes\alpha_3)} & & \downarrow{\scriptstyle O\otimes R} \\
C & \xrightarrow{r(\alpha_4)\,\circ\,r(\alpha_3)} & C''
\end{array}
$$

$$
a_{\alpha_1}a_{\alpha_2}a_{\alpha_3}a_{\alpha_4} =
\begin{pmatrix}
a & p^*m^* & d_2^*d_1^* & \alpha_4^*\alpha_3^*\alpha_2^*\alpha_1^* \\
mp & c & \alpha_4'^*\alpha_3'^*\alpha_2'^*\alpha_1'^* & r_4^*r_3^* \\
d_1d_2 & \alpha_1'\alpha_2'\alpha_3'\alpha_4' & a'' & r^*o^* \\
\alpha_1\alpha_2\alpha_3\alpha_4 & r_3r_4 & or & c''
\end{pmatrix}
\qquad (17)
$$

Since 14 and 17 are the same, condition (h) is satisfied.

It remains now is to show how the union $A^{\alpha_i} \cup A^{\alpha_j}$ is taken. Horizontally: $a_{\alpha_1 \cup \alpha_2} \in A^{\alpha_1} \cup A^{\alpha_2}$ is given by:

$$
\begin{pmatrix}
a & m^* & d_1^* & \alpha_1^* & d_2^*d_1^* & \alpha_2^*\alpha_1^* \\
m & b & \alpha_1'^* & r_1^* & \alpha_2'^*\alpha_1'^* & r_2^*r_1^* \\
d_1 & \alpha_1' & a' & n_1^* & d_2^* & \alpha_2^* \\
\alpha_1 & r_1 & n & b' & \alpha_2'^* & r_2^* \\
d_1d_2 & \alpha_1'\alpha_2' & d_2 & \alpha_2' & a'' & o^* \\
\alpha_1\alpha_2 & r_1r_2 & \alpha_2 & r_2 & o & b''
\end{pmatrix}
\qquad (18)
$$

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Vertically, $a_{\alpha_1 \cup \alpha_3} \in A^{\alpha_1} \cup A^{\alpha_3}$ is given by:

$$
\begin{pmatrix}
a & m^* & p^*m^* & d_1^* & \alpha_1^* & \alpha_3^*\alpha_1^* \\
m & b & p^* & \alpha_1'^* & d_3^* & \alpha_3^* \\
mp & p & c & \alpha_3'^*\alpha_1'^* & \alpha_3'^* & r_3^* \\
d_1 & \alpha_1' & \alpha_1'\alpha_3' & a' & n^* & q^*n^* \\
\alpha_1 & d_3 & \alpha_3' & n & b' & q^* \\
\alpha_1\alpha_3 & \alpha_3 & r_3 & nq & q & c'
\end{pmatrix}
\tag{19}
$$

### 3.1.1 Crossed module algebra

Since a double groupoid with connection is equivalent to a crossed module [5], [19] we should be able to find a crossed module such that the regular representation of its group algebra matches the module structure of the convolution algebra in example 1, our prototype dFB.

For any group $H$, modules over the group ring are crossed $H$-modules where H acts on itself as a normal subgroup of itself by conjugation. $H \times H \to H$. C*$(H \times H) = \mathbb{C}H \times H$. These of course have a representation of bounded linear operators on a Hilbert space, thus via the regular representation we end up with $\mathbb{C}H$ as a module over itself. For example, let $\mathbb{C}H = M_2(\mathbb{C})$. Then we may form the linking algebra 20 (below).

## 3.2 Examples

1. We have already been considering a line bundle $E$ over $G$, a discrete principal double groupoid with folding with one square. We have identified the algebra of sections of $E$ with the convolution algebra of $G$. We reconstruct this example from a '1-Fell bundle' by iterating the categorification procedure.

   The idea for this iteration procedure is based upon Baez's description of nCob where he unpacks (n-1)-dimensional manifolds (objects) redefining them as objects plus morphisms so that what were originally 1-cells become 2-cells [3]. Given a mathematical concept (such as a Fell bundle) once categorified can be categorified again and again until an obvious limit, in the case of nCob, up to n-dimensions. Consider the category 4Cob of 4 dimensional cobordisms whose boundaries are objects which are 3 dimensional manifolds. Consider (roughly) the (weak) 2-category 4Cob$_2$ where 1-morphisms are 3 dimensional cobordisms whose 2 dimensional boundaries are the objects, and the 2-morphisms are cobordisms of those 3 dimensional cobordisms so that the 1-cells in 4Cob are replaced by the 2-cells in 4Cob$_2$. Objects have become 0 and 1-cells, while morphisms have become 1 and 2-cells. Morton has formalised this category in [14] where he defines a double bicategory of cobordisms with corners.

   Described above is *the same* 4-dimensional manifold from two different points of view. The motivation for the 2-category point of view is for quantum gravity

where the 2-morphisms are cobordisms of spacetime whose boundaries are 3-d space, and also in higher gauge theory where 2-morphisms are related to areas swept out by a string. Since we are trying to develop a possible tool to study these physical considerations in an algebraic setting we analogise this iteration process for Fell bundles as our 'space' as follows.

Consider the pair groupoid on 2 objects,

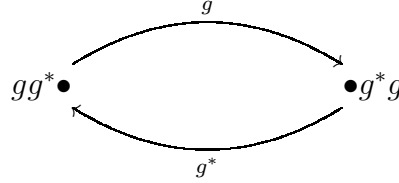$$gg^* \bullet \overset{g}{\underset{g^*}{\rightleftarrows}} \bullet g^*g$$

Figure 6: Pair(2)

and a saturated Fell bundle over it $E$:- a copy of $M_2(\mathbb{C})$ over each element of $G$, denoting the fibres over the unit space: $E_{gg^*}$, $E_{g^*g}$ and the fibres over $g$ and $g^*$ are the Morita equivalence bimodules $E_g$ and $E_{g^*}$ over $E_{gg^*}$ and $E_{g^*g}$. The linking algebra is $M_4(\mathbb{C})$, which defines the algebra of sections of the Fell bundle. The two $C^*$-algebras over the two units are to be the objects of the categorified Fell bundle and the bimodules to be its 1-cells.

As the two $M_2(\mathbb{C})$s are groupoid algebras in their own right, they may be considered as Fell bundle algebras over two more pair groupoids on two objects $\Gamma_1$ and $\Gamma_2$. Thus we iterate the categorification by letting what was a 0-cell become 0 and 1-cells. With this iteration, 2-morphisms arise from $E_g$, and we have formed an example of a double Fell bundle: each 0, 1 and 2-cell is a copy of $\mathbb{C}$; a fibre over each element of the double groupoid. The two groupoids $\Gamma_1$ and $\Gamma_2$ whose algebras are the two copies of $M_2(\mathbb{C})$ plus the elements of the original groupoid, plus new arrows under $E_g$ and $E_{g^*}$ together form a double groupoid: each $g \in G$ is replaced by four arrows because the $E_g$ are also 2-by-2 matrices: one matrix entry for each horizontal arrow in the square, and one for each 2-morphism. There are twice as many of these for a double groupoid as in a double category that does not have inverses. The four arrows directed in the opposite direction arise from $E_{g^*}$ in the same way. The double groupoid and how the 2-morphisms arise from the iteration are shown in the diagrams:

The figure below shows four arrows arising from $E_{g^*}$. The two central arrows are double arrows and the top and bottom arrows are 1-cells in the horizontal category. Recalling that the double arrows are shared by the vertical and horizontal categories, we count 16 double groupoid elements. We feel that for the diagrams we are working with it is conducive to exchange the usual convention for vertical and horizontal categories; in these diagrams the morphisms of that
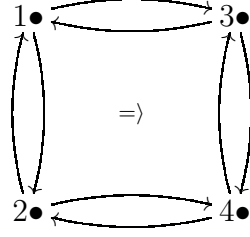
15

Figure 7: double groupoid cell

which is usually referred to as the vertical category are directed across instead of up and down.
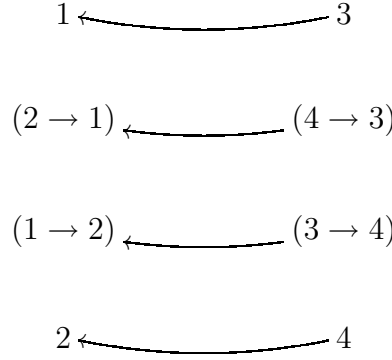

Figure 8: From $E_{g^*}$

and we end up with a line bundle.

If we had begun with a larger algebra, it might have been possible to iterate over and again until arriving at an n-category line bundle.

To clarify, here are the linking algebras discussed above:-

$$\begin{pmatrix} M_{n_1}(\mathbb{C}) & E_{n_1 \times n_2} \\ E_{n_2 \times n_1} & M_{n_2}(\mathbb{C}) \end{pmatrix} \tag{20}$$

where $M_{n_1}(\mathbb{C})$ is the fibre over one of the two groupoid units of $G$, while $M_{n_2}(\mathbb{C})$ is the fibre over the other unit. $E_{n_1 \times n_2}$ and $E_{n_2 \times n_1}$ are $M_{n_1}(\mathbb{C})$-$M_{n_2}(\mathbb{C})$ and $M_{n_2}(\mathbb{C})$-$M_{n_1}(\mathbb{C})$ bimodules respectively, and the Fell bundle algebra is given by the linking algebra. Note that $E$ is saturated. Let $n_1 = n_2 = 2$ Now let $\Gamma_1$ denote the groupoid that $M_{n_1}(\mathbb{C})$ is the algebra of a Fell line bundle over, so that $M_{n_1}(\mathbb{C})$ itself becomes the linking algebra:

$$\begin{pmatrix} E_{g_1 g_1^*} & E_{g_1} \\ E_{g_1^*} & E_{g_1^* g_1} \end{pmatrix} \tag{21}$$

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$$M_{n_1}(\mathbb{C}) = \begin{pmatrix} A & M^* \\ M & B \end{pmatrix}, \quad E_{n_1 \times n_2} = \begin{pmatrix} d^* & \alpha^* \\ \alpha'^* & r^* \end{pmatrix} \tag{22}$$

For example for our prototype line bundle we have linking algebra:

$$\begin{pmatrix} a & m^* & d_1^* & \alpha_1^* \\ m & b & \alpha_1'^* & r_1^* \\ d_1 & \alpha_1' & a' & n^* \\ \alpha_1 & r_1 & n & b' \end{pmatrix} \tag{23}$$

which shows the construction of the algebra we discussed in earlier section (it is the matrix 11).

2. (a) Generalise example 1 slightly by allowing more than one arrow or 2-arrow to have the same domain and range. The algebra of sections is unchanged.

   (b) Allow each fibre to be any simple finite dimensional algebra. As already suggested, the iteration process might be iterated.

3. Take the dual category of a saturated dFB (see 5).

4. Let $\pi : E \to \Gamma$ be a Fell bundle, where $\Gamma$ is a groupoid and $E$ is a small subcategory of Hilb.

5. As Fell bundles are generalisations of C*-algebra bundles and C*-algebraic bundles, dFBs are also in turn generalisations.

6. The obvious generalisations are to consider infinite dimensional algebras over discrete or continuous double groupoids but then it is not possible to use any matrix algebra description. Hitherto, our main motivation is to lay some bricks to build an example of a spectral triple only in finite dimensions.

7. The crossed module with $H \times H \to H$ for discrete group $H$ is an example of an $r$-discrete groupoid. There exists a double groupoid with connection that is equivalent to it ([5]). The convolution algebra should be the algebra of sections of a line dFB (3.1.1).

# 4 Double C*-categories and the Gelfand-Naimark-Segal construction

The Gelfand-Segal-Naimark (GNS) construction was generalised to C*-categories in [8] and we follow suit by taking this generalisation to double C*-categories. We also use this construction to give the homsets the structure of a Hilbert space and consider representations on an internal category in 2Hilb as well as on a 'concrete' double C*-category. A useful reference for the GNS construction is [18].

**Definition: A concrete C\*-category** [8] is a C\*-category $\mathcal{C}$ where objects $A$ are Hilbert spaces (the set of bounded linear maps $(A, A)$ is a C\*-algebra dense in $A$) and the morphisms are bounded linear maps.

**Definition: A concrete double C\*-category** $\mathcal{C}$ is a double C\*-category where the objects are Hilbert spaces and $\mathrm{Ver}_0$ and $\mathrm{Hor}_0$ are concrete C\*-categories. The 2-morphisms are the completed set of bounded linear maps between the Banach spaces that the homsets in the vertical and horizontal categories define.

**Definition: Hilb, 2Hilb and dHilb** [2] Hilb is the category of all Hilbert spaces and bounded linear maps, (thus a concrete C\*-category is a small subcategory of Hilb). 2Hilb is the 2-category of Hilbert spaces and bounded linear maps enriched over Hilb. That is, its homsets are endowed with the structure of a Hilbert space so that 2-morphisms are bounded linear maps between Hilbert spaces. Below we work with a double category dHilb that we define as a small category internal in 2Hilb.

**Definition: A representation of a (double) C\*-category** $\mathcal{U}$ is a (double) \*-functor $F : \mathcal{U} \to \mathcal{C}$. There is also a representation of $\mathcal{U}$ on dHilb, that is, a representation of $\mathcal{U}$ on dHilb is a double \*-functor $\rho : \mathcal{U} \to$ dHilb.

**Definition: cyclic vector**. If A is an object of $\mathcal{U}$, we say that $\xi \in F(A)$ is cyclic for $F$ if

$$\{F(m)\xi : m \in M = (A, B)\}$$

is dense in $F(B)$, and

$$\{F(\alpha)\xi : \alpha \in (M, N), \ d(M) = A, \ d(N) = C\}$$

is dense in $F(C)$.

**Proposition** Let $\mathcal{U}$ be a double C\*-category and $\phi$ a state on the C\*-algebra $(A, A)$. There is a representation $F_\phi$ of $\mathcal{U}$ on a concrete double C\*-category such that:

$$\phi(a) = \langle \xi_\phi, F_{\phi(a)}\xi_\phi \rangle, \ a \in (A, A).$$

If $F_2$ is another representation of $\mathcal{U}$ with cyclic vector $\xi_2 \in F_2(A)$ such that $\phi(a) = \langle \xi_2, F_2(a)\xi_2 \rangle$, $a \in (A, A)$, then there is a unique natural unitary equivalence $u$ between the two representations such that $\xi_\phi = u\xi_2$.

*Proof* Objects of the functor category of $F_\phi$ will be the Hilbert spaces $F_\phi(A)$ for all objects $A$ of $\mathcal{U}$. These are given by the Hilbert space in the usual GNS construction on the C\*-algebra $(A, A)$. To generalise the GNS constuction to double C\*-categories we need to show (a) that there are bounded linear maps $F_\phi(m)$ on the Hilbert space $F_\phi(B)$ and bounded linear maps $F_\phi(\alpha)$ on the Hilbert space $F_\phi(B'')$, (b) there is a

cyclic vector $\xi$ for $F_\phi$ (c) $F_\phi$ is unique up to unitary equivalence (d) $F_\phi$ is a double *-functor.

(a) We begin by defining a semidefinite inner product on $(M, N)$ by

$$\langle \alpha_1, \alpha_2 \rangle = \phi(\alpha_1^* \alpha_2)$$

for all 2-morphisms $\alpha$ with domain $M$ and range $N$.

(It is implicit in the above that we also have a semidefinite inner product on $(A, B)$: $\langle m_1, m_2 \rangle = \phi(m_1^* m_2)$ for all morphisms with domain $A$ and range $B$. In the proof below we can of course always obtain statements by exchanging $\alpha$ for $m$ and domains and ranges $M$ and $N$ for $A$ and $B$.)

Define $F_\phi(\alpha_1) := L_{\alpha_1}$ with $L_{\alpha_1} \alpha_2 := \alpha_1^* \alpha_2$. We are to show that:

$$\| L_{\alpha_1} \| \leq \| \alpha_1 \| \tag{24}$$

where the norm on the left hand side is the operator norm and the norm on the right hand side is the Banach space norm on $(M, N)$. An equivalent assertion is the inequality:

$$\| L_{\alpha_1} \alpha_2 \|^2 \leq \| \alpha_1 \|^2 \| \alpha_2^* \alpha_2 \|^2$$

That is,

$$\phi(\alpha_2^* \alpha_1^* \alpha_1 \alpha_2) \leq \|\alpha_1\|^2 \phi(\alpha_2^* \alpha_2) \tag{25}$$

We see that the above statement is true by virtue of the fact that $\phi$ is a sesquilinear form on $(M, N)$, that is, it satisfies $|\phi(x, y)|^2 \leq \phi(x, x)\phi(y, y)$.

$F_\phi(A)$ is a completion of $(A, A)$, and $L_\alpha$ extends to a unique bounded linear map on $F_\phi(B)$ which we denote $F_\phi(\alpha)$.

(b) $F_\phi(id_A)$ is a cyclic vector $\xi_\phi$ for $F_\phi$ with $\phi(a) = \langle \xi_\phi, F_\phi(a)\xi_\phi \rangle$, $a \in (A, A)$.

(c) Note that $\langle \xi_2, F_2(a)\xi_2 \rangle = \langle \xi_\phi, F_\phi(a)\xi_\phi \rangle$ implies

$$\langle F_2(\alpha_1)\xi_2, F_2(\alpha_2)\xi_2 \rangle = \langle F_\phi(\alpha_1)\xi_\phi, F_\phi(\alpha_2)\xi_\phi \rangle. \tag{26}$$

Thus there is a unitary $u : F_\phi(C) \to F_2(C)$ with $uF_\phi(\alpha)u^{-1} = F_2(\alpha)$ and $u_c\xi_\phi = \xi_2$.

(d) Now we show that $F_\phi$ preserves compositions and identities. For any representation $\pi$ of a unital C*-algebra $A$, $\pi(1) = 1$ where the 1 on the left is the identity in

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$A$ and the 1 on the right is the identity in the Hilbert space that $A$ is represented on. (The C*-algebras in a C*-category have to be unital to satisfy the unit law of category theory.) $F_\phi(id_A) = id_{F_\phi(A)}$. $F_\phi(\alpha_1 \alpha_2) = F_\phi(\alpha_1)F_\phi(\alpha_2)$ by construction of $F_\phi$. Clearly, also $d(F_\phi(\alpha)) = F_\phi(d(\alpha))$ and $r(F_\phi(\alpha)) = F_\phi(r(\alpha))$. So $F_\phi$ is a double functor. Finally we check that it preserves adjoints:

$$\langle F(\alpha)F(a_1), F(a_2)\rangle^* \tag{27}$$
$$= \phi(a_2^*(\alpha a_1)) \tag{28}$$
$$= \phi(\alpha^* a_2)^* a_1) \tag{29}$$
$$= \langle F(a_1), F(\alpha^*)F(a_2)\rangle \tag{30}$$
$$\Rightarrow F(\alpha)^* = F(\alpha^*). \tag{31}$$

**Proposition** By giving $(A, B)$ the structure of a Hilbert space we can repeat the above generalisation of the Gelfand-Naimark-Segal construction for a representation $F_h$ on a dHilb as follows.

*Proof* There is already a positive definite inner product defined on $(A, B)$ and similarly on $(M, N)$, given by $m_1^* m_2$ and $\alpha_1^* \alpha_2$ respectively, but these take values in the C*-algebra $(A, A)$ rather than in $\mathbb{C}$, so it gives the Banach space the structure of a Hilbert module rather than of a Hilbert space. (In the finite dimensional case, this inner product is $k\ \mathrm{tr}(a^* b)$ for $k > 0$.)

First we form the quotient space $V = (A, B)/N_A$ defined by $(m \to m + N_\phi)$ where $N_A$ is the set (ideal of $(A, B)$) of $m \in (A, B)$ for which $\| m \| \neq 0$ but $\phi(m^* m) = 0$. $\phi$ on $V$ is a genuine inner product. We restate the domain of defintion of the bounded linear map $F_h(m)$ to be on $V$ instead of on $(A, B)$ and $L_\alpha$ extends to a bounded linear map on the Hilbert space $(B, B')/N_B$. In the same way as shown above for $F$ on for double C*-category, $F_h$ is a *-functor from $\mathcal{U}$ to dHilb with cyclic vector $\xi_h$ such that

$$\phi(a) = \langle \xi_h, F_h(a)\xi_h\rangle, a \in (A, A).$$

The unitary equivalence is the same as statement (c) in the proof above.

**Observation** A unitary topological quantum field theory (TQFT) is a functor from nCob to Hilb and a 2 dimensional version of a TQFT would define a representation $Z : \mathrm{nCob}_2 \to 2\mathrm{Hilb}$. If a double C*-category $\mathcal{U}$ is viewed as a noncommutative space, then a representation $F_h : \mathcal{U} \to \mathcal{C}$ might lead to an analogue of a TQFT for noncommutative geometry. (If we make $\mathcal{U}$ into a tensor C*-category - by putting tensor product on the set of objects - then $Z$ should reflect the monoidal structure (disjoint union) of nCob.)

# 5 Dual category

In this section we restrict all discussion to saturated (double) Fell bundles.

Since we are motivated to view a dFB as a noncommutative space, we expect it to be useful to define its dual. In particular, we might expect that a finite dimensional dFB over a discrete double groupoid (which is saturated) be closely related to a finite spectral triple. The algebra in a real spectral triple is a tensor product of an algebra $A$ and its opposite algebra $A^o$ [7]. It is with this product algebra that the spectral triple satisfies the reality axiom and the Poincare' duality axiom [6]. The dual of a category whose morphisms are $A$-$B$ bimodules, would be a category whose morphisms are $B$-$A$ bimodules. Equipping a dFB with the Tomita-Takesaki involution $(^o)$ should provide a dual for it.

**Proposition** Let $E$ denote a saturated Fell bundle or double Fell bundle and $A$ its algebra of sections, there is a faithful representation $\rho$ of $A$ on the Hilbert space $H$ in the GNS construction. Since the representation is faithful we won't make a distinction between $A$ and $\rho$. Let $\Gamma$ be an element of $A$, that is, a section of the bundle $E$. Given the action of the involution $^o$ on $E$ defined by $E^o := (\cup E_i)^o = \cup E_i^o$ with $\rho(b)^o \Psi = J\rho(b)^* J\Psi$, $\Psi \in H$, where $J$ is a conjugate linear isometry with $J = J^{-1} = J^*$. Then $E^o$ defines the dual (double) category of $E$. When working in finite dimensions, constructing the dual category amounts to taking the transpose of the sections.

*Proof* Usually when defining dual categories, vector spaces are replaced by their vector space duals and conjugates are taken of morphisms to change their direction. In this case we have a category where morphisms are bimodules over objects so to 'change the direction of the arrows', we turn an $A$-$B$-bimodule into a $B$-$A$ bimodule. And this is exactly the operation that the Tomita-Takesaki involution provides:

Since $(\Gamma b)^o = b^o \Gamma^o$, for the $E_{g^*g}$-$E_{gg^*}$-bimodule $E_g$, $E_g^o$ is an $E_{gg^*}$-$E_{g^*g}$-bimodule. Also note that $E$ is an $A$-$A^o$-bimodule.

For finite dimensional $A$ we can be more explicit as we may write down the Morita equivalence linking algebra. Consider the groupoid Pair(2) with any Fell bundle over it with $m \in E_g$, $a \in E_{gg^*}$, $b \in E_{g^*g}$. An element of the linking algebra is a section:

$$\begin{pmatrix} a & m^* \\ m & b \end{pmatrix} \tag{32}$$

with $E_g \otimes E_g^* \cong E_{gg^*}$ and $E_g^* \otimes E_g \cong E_{g^*g}$.

Now we show that if $J$ is complex conjugation then $J\Gamma^* J = \Gamma^t$ where $t$ denotes transpose.

Let $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and $J\Psi = \bar{\Psi}$ where the bar denotes complex conjugation.

$$b^* J \Psi = \begin{pmatrix} a^* & m^* \\ m & b^* \end{pmatrix} \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix} = \begin{pmatrix} a^* \bar{\psi}_1 + m^* \bar{\psi}_2 \\ m \bar{\psi}_1 + b^* \bar{\psi}_2 \end{pmatrix} \tag{33}$$

$$J b^* J \Psi = \begin{pmatrix} a^t \psi_1 + m^t \psi_2 \\ \bar{m} \psi_1 + b^t \psi_2 \end{pmatrix} = \begin{pmatrix} a^t & m^t \\ \bar{m} & b^t \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = b^t \Psi \tag{34}$$

with $E_g \otimes E_g^* \cong E_{g^*g}$ and $E_g^* \otimes E_g \cong E_{gg^*}$, which means that the imprimitivity bimodules have exchanged directions.

**Example** Consider the double Fell line bundle of example 1 described earlier and figure 2. A section $\Gamma$ of the bundle over one double groupoid square was given by matrix 11. We construct its dual $E^o$ section by section using $J\Gamma J = \Gamma^t$.

$$\Gamma^t = \begin{pmatrix} a^t & m^t & d_1^t & \alpha_1^t \\ \bar{m} & b^t & \alpha_1'^t & r_1^t \\ \bar{d}_1 & \bar{\alpha}_1' & a'^t & n^t \\ \bar{\alpha}_1 & \bar{r}_1 & \bar{n} & b'^t \end{pmatrix} \tag{35}$$

and we illustrate the resulting 2-morphisms as:

$$\begin{array}{ccc}
A & \xleftarrow{\;d\;} & A' \\
{\scriptstyle M}\big\uparrow & {\scriptstyle \alpha} & \big\uparrow{\scriptstyle N} \\
B & \xleftarrow{\;r\;} & B'
\end{array}$$

which is as figure 2 but with the directions of all the arrows reversed.

Thus the dual category $E^o$ is given by the same objects as those of $E$, while the morphisms of $E$, which are Morita equivalence $A$-$B$-bimodules, are replaced with Morita equivalence $B$-$A$-bimodules. The 2-morphisms of $E^o$ are defined by the matrix 35 and the diagram 5. In the case of a concrete double C*-category, the 2-morphisms are the Banach spaces formed from the set of linear maps $(N, M)$. Note that the 2-morphisms are also Morita equivalence bimodules over the domain and range objects of their domain and range Banach space 1-morphisms.

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